

Geometric models of $(d+1)$ -dimensional relativistic rotating oscillators

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Abstract

Geometric models of quantum relativistic rotating oscillators in arbitrary dimensions are defined on backgrounds with deformed anti-de Sitter metrics. It is shown that these models are analytically solvable, deriving the formulas of the energy levels and corresponding normalized energy eigenfunctions. An important property is that all these models have the same nonrelativistic limit, namely the usual harmonic oscillator.

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In general relativity the anti-de Sitter (AdS) spacetime is one of the most important and interesting pieces since the AdS/Conformal field theory-correspondence [1] has been discovered. It is known that, because of the high symmetry of AdS, the free motion of the scalar classical or quantum particles on this background has special features. There is a local chart with a metric [2] able to reproduce the classical motion of an isotropic nonrelativistic harmonic oscillator (NRHO). In other respects, the energy levels of the quantum modes given by the Klein-Gordon equation are equidistant [3]. Thus the geometric models of free test classical or quantum particles on AdS backgrounds represent the relativistic correspondent of the NRHO.

Two years ago, we have generalized this ideal model of relativistic oscillator (RO) to new families of models of RO in $(1+1)$ and $(3+1)$ dimensions

[4, 5]. In general, the metrics of these models are deformations of some AdS or de Sitter metrics that produce oscillations and a specific relativistic rotation effect in the case of the (3+1)-dimensional models [5]. However, what is important here is that all these models have as nonrelativistic limit just the NRHO. We have studied in details the (1+1) [4] models finding that there are two kind of quantum RO with different properties, namely Pöschl-Teller-like models (with deformed AdS metrics) and respectively Rosen-Morse-like models (with deformed de Sitter metrics) [6]. Moreover, we have shown that all the (1+1)-dimensional Pöschl-Teller-like RO have similar properties and the same $so(1, 2)$ dynamical algebra [7]. The next step might be the study of the algebras of the (3+1) RO for which we have obtained only the energy spectra and the wave functions up to normalization factors [5]. Fortunately, the space dimension of the model is not determinant for solving the Klein-Gordon equation. This means that these models can be analytically solved in any dimensions like the AdS one [8]. Thus we have the opportunity to completely solve the problem of the quantum modes of rotating RO in arbitrary dimensions and then turn to the study of their dynamical algebras.

Our aim is to present here only the method of solving the Pöschl-Teller-like models in (d+1) dimensions. Our main objective is to derive the formula of the energy levels and to find the normalized energy eigenfunctions in spherical coordinates (and natural units with $\hbar = c = 1$). Moreover, we show that, like in (3+1) dimensions, the rotation of these (d+1) RO is a pure relativistic effect that vanishes in the nonrelativistic limit when all these models lead to the (d+1)-dimensional NRHO.

From the theory of the (3+1) RO we understand that the background of any (d+1) RO must be static and spherically symmetric (central) [4]. Therefore, the backgrounds of our (d+1) RO must have central static charts with generalized spherical coordinates, $r, \theta_1, \dots, \theta_{d-1}$, commonly related with the Cartesian ones $\mathbf{x} \equiv (x^1, x^2, \dots, x^d)$ [10]. Here it is convenient to chose the radial coordinate such that $g_{rr} = -g_{00}$ since then the radial scalar product is simpler [7]. Starting with these options, we define the metrics of our (d+1)-dimensional Pöschl-Teller-like RO as one-parameter *deformations* of the AdS metric given by the line elements

$$ds^2 = \left(1 + \frac{1}{\epsilon^2} \tan^2 \hat{\omega} r\right) (dt^2 - dr^2) - \frac{1}{\hat{\omega}^2} \tan^2 \hat{\omega} r d\theta^2 \quad (1)$$

where we denote $\hat{\omega} = \epsilon \omega$, $\epsilon \in [0, \infty)$, and

$$d\theta^2 = d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 \dots + \sin^2 \theta_1 \sin^2 \theta_2 \dots \sin^2 \theta_{d-2} d\theta_{d-1}^2 \quad (2)$$

is the usual line element on the sphere S^{d-1} . The deformation parameter ϵ determines the geometry of the background while ω remains fixed. It is clear that for $\epsilon = 1$ we obtain just the AdS metric (with the hyperboloid radius $R = 1/\omega$) [8]. An interesting case is that of $\epsilon \rightarrow 0$ when the line element,

$$ds^2 = (1 + \omega^2 r^2)(dt^2 - dr^2) - r^2 d\theta^2, \quad (3)$$

defines a background where the relativistic quantum motion is similar to that of NRHO. This model will be called the *normal* RO. In general, the radial domain of any RO is $D_r = [0, \pi/2\hat{\omega})$ which means that the whole space domain is $D = D_r \times S^{d-1}$. For the models with $\epsilon \neq 0$ the time might satisfy the condition $t \in [-\pi/\hat{\omega}, \pi/\hat{\omega})$ as in the AdS case but here we consider that $t \in (-\infty, \infty)$ which corresponds to the universal covering spacetimes of the original backgrounds.

In our models the oscillating test particle is described by a scalar quantum field ϕ of mass M , minimally coupled with the gravitational field. Its quantum modes are given by the particular solutions of the Klein-Gordon equation

$$\frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} g^{\mu\nu} \partial_\nu \phi) + M^2 \phi = 0, \quad g = |\det(g_{\mu\nu})|. \quad (4)$$

These may be either square integrable functions or distributions on D . In both cases they must be orthonormal (in usual or generalized sense) with respect to the relativistic scalar product [9]

$$\langle \phi, \phi' \rangle = i \int_D d^d x \sqrt{g} g^{00} \phi^* \overleftrightarrow{\partial}_0 \phi'. \quad (5)$$

The spherical variables of Eq.(4) can be separated by using generalized spherical harmonics, $Y_{l(\lambda)}^{d-1}(\mathbf{x}/r)$. These are normalized eigenfunctions of the angular Laplace operator [10],

$$-\Delta_S Y_{l(\lambda)}^{d-1}(\mathbf{x}/r) = l(l + d - 2) Y_{l(\lambda)}^{d-1}(\mathbf{x}/r), \quad (6)$$

corresponding to eigenvalues depending on the *angular* quantum number l which takes the values $0, 1, 2, \dots$ [10]. The notation (λ) stands for a collection

of quantum numbers giving the multiplicity of these eigenvalues [10],

$$\gamma_l = (2l + d - 2) \frac{(l + d - 3)!}{l! (d - 2)!}. \quad (7)$$

We start with (positive frequency) particular solutions of energy E ,

$$\phi_{E,l(\lambda)}^{(+)}(t, \mathbf{x}) \sim (\cot \hat{\omega} r)^{\frac{d-1}{2}} R_{E,l}(r) Y_{l(\lambda)}^{d-1}(\mathbf{x}/r) e^{-iEt}. \quad (8)$$

Then, after a few manipulation, we find the radial equation

$$\left[-\frac{1}{\hat{\omega}^2} \frac{d^2}{dr^2} + \frac{2s(2s-1)}{\sin^2 \hat{\omega} r} + \frac{2p(2p-1)}{\cos^2 \hat{\omega} r} \right] R_{E,l} = \nu^2 R_{E,l} \quad (9)$$

where

$$2s(2s-1) = \left(l + \frac{d}{2} - 1 \right)^2 - \frac{1}{4}, \quad 2p(2p-1) = \frac{M^2}{\hat{\omega}^2 \epsilon^2} + \frac{d^2 - 1}{4}, \quad (10)$$

and

$$\nu^2 = \frac{E^2}{\hat{\omega}^2} - \left(1 - \frac{1}{\epsilon^2} \right) \left[\frac{M^2}{\hat{\omega}^2} - l(l + d - 2) \right]. \quad (11)$$

This equation gives radial functions,

$$R_{E,l}(r) \sim \sin^{2s} \hat{\omega} r \cos^{2p} \hat{\omega} r F \left(s + p - \frac{\nu}{2}, s + p + \frac{\nu}{2}, 2s + \frac{1}{2}, \sin^2 \hat{\omega} r \right), \quad (12)$$

expressed in terms of Gauss hypergeometric functions [11] depending on the real parameters s , p and ν . The radial functions (12) have good physical meaning only when the functions F are polynomials selected by a suitable quantization condition since otherwise these are strongly divergent for $r \rightarrow \pi/2\hat{\omega}$. Therefore, we introduce the radial quantum number n_r [12] and impose the quantization condition

$$\nu = 2(n_r + s + p), \quad n_r = 0, 1, 2, \dots \quad (13)$$

In addition, we choose the boundary conditions of the *regular* modes given by the positive solutions of Eqs.(10), i.e. $2s = l + (d-1)/2$ and $2p = k - (d-1)/2$, where we denote

$$k = \sqrt{\frac{M^2}{\hat{\omega}^2 \epsilon^2} + \frac{d^2}{4}} + \frac{d}{2}. \quad (14)$$

This new parameter which concentrates all the other ones can be used as the main parameter of RO, like in the case of the (1+1) models where k was just the weight of the irreducible representations of the $so(1, 2)$ dynamical algebra [7].

The last step is to define the *main* quantum number, $n = 2n_r + l$, which takes the values, $0, 1, 2, \dots$, giving the energy levels

$$E_{n,l}^2 = \hat{\omega}^2(k+n)^2 + \hat{\omega}^2(\epsilon^2 - 1) \left[k(k-d) - \frac{1}{\epsilon^2} l(l+d-2) \right]. \quad (15)$$

If n is even then $l = 0, 2, 4, \dots, n$ while for odd n we have $l = 1, 3, 5, \dots, n$. In both cases the degree of degeneracy of the level $E_{n,l}$ is given by (7). The last term of (15) is just the rotator-like term that gives the behavior of rotating oscillator. Obviously, this does not contribute in the AdS case when $\epsilon = 1$. On the other hand, we observe that the rotation effect vanishes in the nonrelativistic limit since the rotator-like term decreases as $1/c^2$ when $c \rightarrow \infty$ (in usual units). Now it remains only to express (12) in terms of Jacobi polynomials and to normalize them to unity with respect to (5). The final result is

$$\begin{aligned} \phi_{n,l(\lambda)}^{(+)}(t, \mathbf{x}) &= N_{n,l} \sin^l \hat{\omega} r \cos^k \hat{\omega} r \\ &\times P_{n_r}^{(l+\frac{d}{2}-1, k-\frac{d}{2})}(\cos 2\hat{\omega} r) Y_{l(\lambda)}^{d-1}(\mathbf{x}/r) e^{-iE_{n,l}t}, \end{aligned} \quad (16)$$

where

$$N_{n,l} = \left[\frac{\hat{\omega}^d}{E_{n,l}} \frac{n_r! (2n_r + k + l) \Gamma(n_r + k + l)}{\Gamma(n_r + l + \frac{d}{2}) \Gamma(n_r + k + 1 - \frac{d}{2})} \right]^{\frac{1}{2}}. \quad (17)$$

Particularly, for $\epsilon = 1$ we recover the result we have recently obtained in AdS case [8].

According to the above result, we can say that in the models with $\epsilon \neq 0$ the particles have the same space behavior. However, the situation is different for $\epsilon \rightarrow 0$. It is not difficult to show that in this limit, when k increases as $M/\omega\epsilon^2$, we obtain the energy levels of the normal RO

$$\lim_{\epsilon \rightarrow 0} E_{n,l}^2 = \overset{\circ}{E}_{n,l}^2 = M^2 + 2\omega M \left(n + \frac{d}{2} \right) + \omega^2 l(l+d-2) \quad (18)$$

which have their specific rotator-like terms (of the order $1/c^2$). The corresponding energy eigenfunctions can be expressed in terms of Laguerre poly-

nomials as,

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \phi_{n,l(\lambda)}^{(+)}(t, \mathbf{x}) &= \overset{o}{\phi}_{n,l(\lambda)}^{(+)}(t, \mathbf{x}) = \left[\frac{(\omega M)^{l+\frac{d}{2}}}{\overset{o}{E}_{n,l}} \frac{n_r!}{\Gamma(n_r + l + \frac{d}{2})} \right]^{\frac{1}{2}} \\ &\times r^l e^{-\omega M r^2/2} L_{n_r}^{l+\frac{d}{2}-1}(\omega M r^2) Y_{l(\lambda)}^{d-1}(\mathbf{x}/r) e^{-i \overset{o}{E}_{n,l} t}. \end{aligned} \quad (19)$$

It is remarkable that these wave functions coincide to those of NRHO up to the factor $1/\sqrt{2 \overset{o}{E}_{n,l}}$ which appears since the definition of the relativistic scalar product is different from that of the nonrelativistic one. Moreover, one can verify that in the nonrelativistic limit, (for $c \rightarrow \infty$ and very small nonrelativistic energies $\tilde{E} = E - Mc^2$, in usual units), the energy levels (18) become just those of the NRHO, i.e. $\tilde{E}_n = \omega(n + d/2)$. On the other hand, we observe that the nonrelativistic limit of any RO can be calculated taking first $\epsilon \rightarrow 0$ and then $c \rightarrow \infty$. Therefore, we can conclude that all the models of rotating RO studied here lead to the (d+1)-dimensional NRHO in the nonrelativistic limit.

References

- [1] J. Maldacena, *preprint hep-th/971120*
- [2] E. van Beveren, G. Rupp, T. A. Rijken, C. Dullemond, *Phys. Rev. D* **27**, 1527 (1983); C. Dullemond, E. van Beveren, *Phys. Rev. D* **28**, 1028 (1983)
- [3] S. J. Avis, C. J. Isham and D. Storey, *Phys. Rev.* **D10**, 3565 (1978)
- [4] I. I. Cotăescu, *Int. J. Mod. Phys. A* **12**, 3545 (1997)
- [5] I. I. Cotăescu, *Mod. Phys. Lett. A* **12**, 685 (1997)
- [6] I. I. Cotăescu, *Theor. Math. Comp. Phys.*¹ **1**, 15 (1998)
- [7] I. I. Cotăescu and G. Drăgănescu, *J. Math. Phys.* **38**, 5505 (1997); I. I. Cotăescu, *J. Math. Phys.* **39**, 3043 (1998)

¹ New series of *Annals of the West University of Timișoara*

- [8] I. I. Cotăescu, to be published in *Phys. Rev.* **D**
- [9] N. D. Birrel and P. C. W. Davies, *Quantum Field in Curved Space*, Cambridge University Press, Cambridge (1982)
- [10] M. E. Taylor, *Partial Differential Equations*, Springer Verlag, N.Y. (1996)
- [11] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* Dover (1964)
- [12] C. P. Burgess and C. A. Lütken, *Phys. Lett.* **153B**, 137 (1985)